Electric charge in interaction with magnetically charged black holes

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ABSTRACT: We examine the angular momentum of an electric charge e placed at rest outside a dilaton black hole with magnetic charge Q. The electromagnetic angular momentum which is stored in the electromagnetic field outside the black hole shows several common features regardless of the dilaton coupling strength, though the dilaton black holes are drastically different in their spacetime structure depending on it. First, the electromagnetic angular momentum depends on the separation distance between the two objects and changes monotonically from eQ to 0 as the charge goes down from infinity to the horizon, if rotational effects of the black hole are discarded. Next, as the black hole approaches extremality, however, the electromagnetic angular momentum tends to be independent of the distance between the two objects. It is then precisely eQ as in the electric charge and monopole system in flat spacetime. We discuss why these effects are exhibited and argue that the above features are to hold in widely generic settings including black hole solutions in theories with more complicated field contents, by addressing the no hair theorem for black holes and the phenomenon of field expulsion exhibited by extremal black holes.

KEYWORDS: Black Holes, Black Holes in String Theory.

Contents

1. Introduction

The influence of a gravitational field on external fields around black holes was investigated in 1970s and 1980s by a number of authors. In particular, the electrostatic field near a black hole was derived by Hanni and Ruffini [1] for a point charge at rest, by Cohen and Wald [2] for multipole fields, and by Linet [3] in algebraic form in Schwarzschild spacetime. Leaute and Linet [4] applied this test field approximation to the case of a Reissner-Nordström black hole background. Recently, the problem of a point charge at rest near a Reissner-Nordström black hole was reconsidered by Bini, Geralico and Ruffini [5, 6] using the first order perturbation approach formulated by Zerilli [7]. External stationary magnetic fields were also considered in the presence of Kerr black holes using the test field approximation by Wald [8] and of Reissner-Nordström black holes employing the first order coupled perturbations by Bičák and Dvořák [9]. The authors found that, as electromagnetic multipole moments approach the horizon of a black hole, all the multipole moments fade away except the monopole. That is, the no hair conjecture for the electromagnetic multipole moments holds for the black hole and so there is no black hole with multipole moments other than the monopole. Another noticeable result of these works is that the component of the electromagnetic field normal to the horizon tends to vanish as the black holes approach extremality and the flux lines are totally expelled in the extremal limit. This is analogous to the "Meissner effect" for a magnetic field around a superconductor [10].

In this paper we consider a system of an electric charge and a magnetically charged dilaton black hole (instead of an electrically charged Reissner-Nordström black hole). We don't try to derive the electric field structure of the charge. Instead, we examine the angular momentum associated with the system. As is well known, the angular momentum of a static configuration of an electric charge e and a magnetic monopole Q is precisely eQ in

flat space [11]. The angular momentum is all stored in the electromagnetic fields. However, when the monopole is replaced with a magnetically charged black hole, the electromagnetic angular momentum[1] changes in a somewhat complicated way. Such system was considered originally by Garfinkle and Rey [12] many years ago and by Bunster and Henneaux [22] recently. As was shown by Garfinkle and Rey, for a system of an electric charge e held at rest outside a magnetically charged black hole of strength Q in the Einstein-Maxwell theory, the angular momentum stored in the electromagnetic field outside the black hole does depend upon the separation distance between the two objects. The electromagnetic angular momentum changes monotonically from 0 to eQ as the charge moves from the horizon to infinity. However, in the extreme limit of the black hole it does not depend on the separation and is precisely eQ, as in the system of an electric charge and a monopole in flat space.

We shall see below that these results remain to be true even when the theory includes dilatonic couplings. This is somewhat striking because charged black hole solutions of the dilaton gravity [13, 14] have many properties distinct from those of the Reissner-Nordström black hole, depending on the strength of the dilaton coupling α. Each dilaton black hole has only one horizon and a spacelike singularity giving rise to a Schwarzschild-type space-time structure, whereas the typical Reissner-Nordström black hole has two horizons and a timelike singularity. Furthermore, the event horizon of a dilaton black hole is actually singular in the extremal limit and has zero area, as opposed to that of the Reissner-Nordström black hole which has finite area and is nonsingular. The thermodynamical relationships also differ depending on the dilaton coupling α which is a nonnegative constant. In the extremal limit the dilaton black holes have zero entropy and their temperature is zero for $\alpha < 1$, finite for $\alpha = 1$, and infinite for $\alpha > 1$. Extremal black holes for $\alpha > 1$ act more like elementary particles than usual Reissner-Nordström black holes because infinite potential barriers form outside the horizon [16]. Many other features of the dilaton black hole solutions (but not all) depend on the coupling strength α. In spite of these distinctive differences, however, the qualitative features of the electromagnetic angular momentum are not modified in the presence of the magnetically charged dilaton black hole (instead of the Reissner-Nordström black hole). Regardless of the strength of the dialton coupling, the electromagnetic angular momentum also changes monotonically from 0 to eQ as the charge moves from the horizon to infinity for non-extreme dilaton black holes, while it is independent of the separation distance with the precise value eQ in the extremal limit of the black hole.

Since the dilaton black holes are drastically different in their spacetime structure from the Reissner-Nordström black holes, the fact that qualitative features of the electromagnetic angular momentum are independent of the dialton coupling makes us expect that such features are common to black holes in various theories with more complicated field contents. Indeed, the nature of the electromagnetic angular momentum seems to be generic for various black holes as far as the no hair theorem holds for the black holes and electro-

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[1]We will use this term to designate the angular momentum stored in the electromagnetic field outside the black hole.

magnetic fields are expelled from the horizon in the extremal limit of the black holes. We shall argue that the separation dependence of the electromagnetic angular momentum in the non-extremal case is related to the nontrivial spacetime structure, i.e., the existence of the non-degenerate horizon, and the no hair nature of black holes. On the other hand, the fact that it is independent of the separation distance and its value is precisely eQ in the extremal limit comes from the phenomenon of flux expulsion exhibited by extreme black holes mentioned in the first paragraph of this section.

The plan of our work is as follows. In the next section, we shall first summarize the magnetically charged black hole solution to the Einstein-Maxwell-dilaton gravity. In section 3, we derive an equation governing the angular momentum function of the system of an electric charge and a magnetic dilaton black hole by extending Garfinkle and Rey's work. In section 4, we analyze solutions of the equation. In section 5, we examine the electromagnetic angular momentum of the charge and black hole system analytically and numerically. In the last section, we argue that the monotonic variation of the electromagnetic angular momentum from 0 to eQ in the non-extremal black hole case comes from the no hair nature of the black hole and that its independence from the separation distance in the extremal limit of black holes results from the flux expulsion by extremal black holes.

2. Magnetically charged dilaton black holes

In this section we give a brief introduction to the charged dilaton black holes in $3 + 1$ dimensions found by Gibbons et al.[13] and Garfinkle et al.[14] and slowly rotating dilaton black holes discussed by Horne et al.[15]. The action for the Einstein-Maxwell-dilaton gravity is given by, in the Einstein frame,

$$S = \int d^4x \sqrt{-g} \left[-\mathrm{R} + 2(\nabla\phi)^2 + e^{-2\alpha\phi} F^2 \right],$$
(2.1)

where R is the scalar curvature, ϕ the dilaton field and $F_{\mu\nu}$ the Maxwell field strength. α is a non-negative constant governing the strength of the dilaton coupling to the Maxwell field. For $\alpha = 0$ this action corresponds to the standard Einstein-Maxwell action with an extra free scalar field. When $\alpha = 1$, this action describes a part of the tree-level low energy limit of string theory in the Einstein frame. The case $\alpha = \sqrt{3}$ gives simply the Kaluza-Klein action which is obtained by dimensionally reducing the five-dimensional vacuum Einstein action. The equations of motion derived from the above action (2.1) are

$$\nabla_a(e^{-2\alpha\phi} F^{ab}) = 0,$$
(2.2)

$$\nabla^2\phi + \frac{\alpha}{2} e^{-2\alpha\phi} F^2 = 0,$$
(2.3)

$$\mathrm{R}_{ab} = 2\nabla_a\phi\nabla_b\phi + 2e^{-2\alpha\phi} F_{ac}F_b{}^c - \frac{1}{2} g_{ab}e^{-2\alpha\phi} F^2.$$
(2.4)

The magnetically charged and spherically symmetric static black hole solution of the

above equations takes the form [13, 14]

$$F = Q \sin\theta \; d\theta \wedge d\varphi, \tag{2.5}$$

$$e^{-2\phi} = \left(1 - \frac{r_-}{r}\right)^{2\alpha/(1+\alpha^2)}, \tag{2.6}$$

$$ds^2 = -\lambda^2 dt^2 + \frac{dr^2}{\lambda^2} + R^2 d\Omega^2, \tag{2.7}$$

where

$$\lambda^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-\alpha^2)/(1+\alpha^2)}, \tag{2.8}$$

$$R^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{2\alpha^2/(1+\alpha^2)}. \tag{2.9}$$

Here, r_+ and r_- are the values of the parameter r at the outer and the inner horizon, respectively, and are related to the physical mass and the magnetic charge by

$$M = \frac{r_+}{2} + \left(\frac{1-\alpha^2}{1+\alpha^2}\right)\frac{r_-}{2}, \tag{2.10}$$

$$Q = \left(\frac{r_+ r_-}{1+\alpha^2}\right)^{1/2}. \tag{2.11}$$

R, not r, has the normal meaning of the radial variable, in the sense that the area of the sphere obtained by varying θ and φ at fixed t and r (or R) is $4\pi R^2$.

The qualitative features of dilaton black holes depend crucially on α. When $\alpha = 0$, this solution reduces to the Reissner-Nordström solution of the Einstein-Maxwell theory. The geometry is then not singular at either of r_- and r_+. For all α, the surface $r = r_+$ is an event horizon. However, for $\alpha \neq 0$ the geometry is singular at r_-, because R vanishes there. In fact, the surface $r = r_-$ is a spacelike curvature singularity very similar to that in the Schwarzschild metric. The singularity is shielded by the horizon if $M \geq M_c \equiv Q/\sqrt{1+\alpha^2}$. Due to this difference, there occurs a big difference in the spacetime structure of the extremal black holes in the two cases. For $\alpha = 0$ the area of the event horizon is finite. However, for $\alpha \neq 0$ the area vanishes for the extremal black holes and the geometry is singular there.

The temperature and entropy of the black holes are given by, respectively,

$$T = \frac{1}{4\pi r_+}\left(1 - \frac{r_-}{r_+}\right)^{(1-\alpha^2)/(1+\alpha^2)}, \tag{2.12}$$

$$S = \pi r_+{}^2 \left(1 - \frac{r_-}{r_+}\right)^{2\alpha^2/(1+\alpha^2)}. \tag{2.13}$$

Evidently the extremal black holes will have finite entropy in case $\alpha = 0$, but zero entropy otherwise. The temperature of the extremal black holes is zero for $\alpha < 1$, finite and equal to $1/8\pi M$ for $\alpha = 1$, and infinite for $\alpha > 1$. Extremal black holes with non-zero temperature inevitably develop naked singularities and are therefore not physically acceptable. However, Holzhey and Wilczek [16] showed that black holes with $\alpha > 1$ have

infinite mass gaps. These black holes can neither absorb any finite energy from impinging objects nor radiate into these modes. In these senses the $\alpha > 1$ extremal black holes act more like elementary particles.

Now we brief on the rotating dilaton black hole solution for further works in the following sections. Since there is no explicit solution describing rotating charged black hole for arbitrary α, the rotating solution is considered in the limit of slow rotation of [15]. $g_{t\phi}$ is the only term in the metric that changes to first order in the angular momentum parameter a. The dilaton does not change to $O(a)$, and A_t is the only component of the vector potential that changes to $O(a)$. For arbitrary α, in the limit of slow rotation the metric has the following form:

$$ds^2 = -\lambda^2(r)dt^2 + \frac{dr^2}{\lambda^2(r)} + R^2(r)d\Omega^2 - 2af(r)R^2(r)\sin^2\theta dt d\phi, \qquad (2.14)$$

where

$$f(r) \equiv \frac{(1+\alpha^2)^2}{(1-\alpha^2)(1-3\alpha^2)}\frac{1}{r_-^2} \qquad (2.15)$$
$$- \left[\frac{1}{r^2} + \frac{(1+\alpha^2)^2}{(1-\alpha^2)(1-3\alpha^2)}\frac{1}{r_-^2} + \frac{1+\alpha^2}{1-\alpha^2}\frac{1}{r_-r} - \frac{r_+}{r^3} \right]\left(1 - \frac{r_-}{r}\right)^{(1-3\alpha^2)/(1+\alpha^2)}.$$

The surface gravity and the area of the event horizon do not change to $O(a)$. However, for $\alpha > 1$, a small amount of rotation produces a large change in the geometry close to the horizon of a nearly extremal black hole. In the extremal limit, as r approaches $r_+ = r_-$, $f(r)R^2(r)$ diverges for $\alpha > 1$. The angular momentum changes to $O(a)$ as discussed in [15]. The angular momentum J is related to the asymptotic form of the metric by

$$g_{t\varphi} \sim -\frac{2J}{r}\sin^2\theta + O\left(\frac{1}{r^2}\right), \qquad (2.16)$$

which gives

$$J = \frac{a}{6}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right). \qquad (2.17)$$

3. Equation for the angular momentum

In this section we derive an equation governing the angular momentum of a particle of mass m and charge e at rest in the field of a magnetically charged dilaton black hole. We assume that the charge and mass of the particle is small enough and hence we can calculate the angular momentum to first order in m and e. It is straightforward to solve Einstein-Maxwell-dilaton equations eqs. (2.2)–(2.4) to first order, because most of components of the metric and the gauge field change only to second order. Though many of the interesting physical quantities come into second order, we can still extract some useful information on the angular momentum from the first order analysis.

To zeroth order the fields and the metric are described by eq. (2.5)–(2.7). Since the system is axisymmetric, the components of the metric and the gauge field that come into

first order can be deduced from the inspection of axially symmetric black holes like the Kerr-Newman solution. The only terms of the metric and the gauge field that change to first order are $g_{t\varphi}$ and A_t. The existence of A_t also follows from the Bianchi identity $\nabla_{[a}F_{bc]} = 0$ and the fact that the Lie derivative of F_{ab} with respect to $(\partial/\partial t)^a$ vanishes. The corrections to other components of the fields and the metric come into second order, and can be ignored in the present analysis. Note that the equations of motion for the dilaton receive no correction to first order and thus the behavior of the dilaton is unchanged from eq. (2.6). Therefore, we need only to develop a first order equation in $g_{t\varphi}$ and A_t to get information for the angular momentum. We do this by extending straightforwardly Garfinkle and Rey's work for a magnetically charged Reissner-Nordstöm black hole to the case of a magnetically charged dilaton black hole.

Let the point electric charge be at rest at a point $r = b$ on the polar axis $\theta = \pi$ in the field of a magnetically charged dilaton black hole, then its stress-energy tensor and current vector are given by

$$T^{ab} = \frac{m}{2\pi R^2(b)\sin\theta}\,\lambda(b)^{-1}\,\delta(r-b)\,\delta(\theta-\pi)\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b, \tag{3.1}$$

$$j^a = \frac{e}{2\pi R^2(b)\sin\theta}\,\delta(r-b)\,\delta(\theta-\pi)\left(\frac{\partial}{\partial t}\right)^a. \tag{3.2}$$

The system is stationary and axisymmetric, and thus has its corresponding Killing vectors $(\partial/\partial t)^a$ and $(\partial/\partial\phi)^a$. In an axisymmetric, asymptotically flat spacetime with the axial Killing vector $(\partial/\partial\phi)^a$, the total angular momentum [19] J can be calculated by performing the Komar integral defined by

$$J \equiv \frac{1}{16\pi}\int_S \varepsilon_{abcd}\nabla^c\left(\frac{\partial}{\partial\varphi}\right)^d, \tag{3.3}$$

where S is a two-sphere in the asymptotic region where T_{ab} is assumed to vanish. This total angular momentum of the system contains both the intrinsic spin of the black hole and the angular momentum stored in the electromagnetic field. To extract directly the information on the angular momentum from the equations of motion, we define the angular momentum contained within a two-sphere $S(r)$ of constant r and t:

$$\mathcal{L}(r) \equiv \frac{1}{16\pi}\int_{S(r)} \varepsilon_{abcd}\nabla^c\left(\frac{\partial}{\partial\varphi}\right)^d, \tag{3.4}$$

and the total angular momentum J is then given by

$$J = \lim_{r\to\infty}\mathcal{L}(r). \tag{3.5}$$

Obtaining the equation governing $\mathcal{L}(r)$ is the purpose of this section. $\mathcal{L}(r)$ can be expressed in terms of $g_{t\varphi}$ and unperturbed metrics of the black hole. By straightforwardly calculating the integral of eq. (3.4), we obtain the following expression for the angular momentum to first order in $g_{t\varphi}$

$$\mathcal{L}(r) = \frac{1}{6}R^4\frac{d}{dr}\left(\frac{\chi}{R^2}\right), \tag{3.6}$$

where R^2 is the radial metric component of (2.9) and χ is defined as follows

$$\chi \equiv \frac{3}{4} \int_0^\pi g_{t\varphi} \sin\theta \ d\theta. \tag{3.7}$$

Since $g_{t\varphi}$ and A_t appear mixed in the Einstein and Maxwell equations, it will be convenient to define another quantity u related to A_t by

$$u \equiv -\frac{3}{2} \int_0^\pi A_t \sin\theta \cos\theta d\theta. \tag{3.8}$$

Taking the $t-\varphi$ component of Einstein equation (2.4) and the t component of Maxwell equation (2.2), evaluating all quantities to first order in $g_{t\varphi}$ and A_t, and using the definitions (3.7) and (3.8), we find the following equations for u and χ:

$$\lambda^2 \frac{d^2\chi}{dr^2} - \frac{2}{R^2}\left(1 - \frac{1}{2}\frac{dR^2}{dr}\frac{d\lambda^2}{dr} + Q^2\frac{e^{-2\alpha\phi}}{R^2}\right)\chi + 4Q\frac{e^{-2\alpha\phi}}{R^2}u = 0, \tag{3.9}$$

$$\frac{\lambda^2}{R^2}\frac{d}{dr}\left(R^2\frac{du}{dr}\right) + \lambda^2\frac{r^2}{R^2}\frac{d}{dr}\left(\frac{R^2}{r^2}\right)\frac{du}{dr} - \frac{2u}{R^2} + \frac{2Q}{R^4}\chi = 6\pi e^{2\alpha\phi}\int_0^\pi j_t \sin\theta\cos\theta d\theta. \tag{3.10}$$

Inserting eqs. (2.6), (2.8) and (3.6) into eq. (3.9), u can be expressed in terms of $\mathcal{L}$ as follows

$$u = -\frac{3}{2Q}\frac{r^2}{R^2}\lambda^2\frac{d\mathcal{L}}{dr} + \frac{Q}{R^2}\chi. \tag{3.11}$$

Finally, inserting this expression into eq. (3.10), we get the following equation which governs the angular momentum function $\mathcal{L}(r)$:

$$\frac{d}{dr}\left[\frac{R^4}{r^2}\frac{d}{dr}\left(\frac{\lambda^2 r^2}{R^2}\frac{d\mathcal{L}}{dr}\right) - 2\left(1 + \frac{2Q^2}{r^2}\right)\mathcal{L}\right] = -\frac{4\pi Q R^2}{\lambda^2}\int_0^\pi j_t \sin\theta\cos\theta d\theta$$

$$= -2eQ \ \delta(r - b), \tag{3.12}$$

where we have used eq. (3.2) to evaluate the integral. eq. (3.12) is the main result of this section. It is easy to check that, when $\alpha = 0$, eq. (3.12) reduces to the equation for $\mathcal{L}$ obtained in ref. [12].

4. Solutions of the second-order equation

Since the right hand side of eq. (3.12) vanishes when $r \neq b$, it will be convenient to consider the nonhomogeneous, linear, second-order differential equation

$$\mathcal{D}\mathcal{L} = C, \tag{4.1}$$

where C is an integration constant and $\mathcal{D}$ is the differential operator defined as

$$\mathcal{D} \equiv \frac{R^4}{r^2}\frac{d}{dr}\left(\frac{\lambda^2 r^2}{R^2}\frac{d}{dr}\right) - 2\left(1 + \frac{2Q^2}{r^2}\right). \tag{4.2}$$

It follows from eq. (3.12) that C is discontinuous at $r = b$ and the difference is given by

$$C_> - C_< = -2eQ \qquad (4.3)$$

where $C_>$ and $C_<$ are values of $\mathcal{DL}$ in regions $r > b$ and $r < b$ respectively. The constants $C_>$ and $C_<$ may depend on b but not on r.

The general solution of the second-order equation (4.1) is composed of two homogeneous solutions and a particular solution, and satisfies automatically the third-order equation (3.12) when $r \neq b$. A particular solution $\ell_p(r)$ of the inhomogeneous equation is given by:

$$\ell_p(r) = \frac{a}{6}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_- - \frac{4}{1+\alpha^2}\frac{r_+r_-}{r}\right), \qquad (4.4)$$

with the inhomogeneous term

$$C = -\frac{a}{3}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right). \qquad (4.5)$$

This inhomogeneous solution behaves well in the whole region of our interest, from r_+ to ∞. As we will see in the next section, this corresponds to the angular momentum of the slowly rotating dilaton black hole.

On the other hand, we have no explicit solutions of the second-order homogeneous equation (4.1) with $C = 0$ for arbitrary α. However, the properties of two linearly independent solutions of the homogeneous equations can be inferred by examining the indicial equations near the singularities at $r = r_+$ and ∞. For non-extremal black holes (as $r_+ > r_-$), the indicial equations say that one solution approaches a finite value, but the other solution diverges in the order of $\ln(r - r_+)$ at $r = r_+$. As r goes to the infinity, i.e., $r \to \infty$, one solution diverges in the order of r^2 but the other solution converges to zero in the order of r^{-1}.

Furthermore, it can be easily shown that there can be no homogeneous solution finite at both the singularities, i.e., one solution which is finite at r_+ should diverge at ∞ and, on the contrary, the other solution which is divergent at r_+ should be zero at ∞. The proof is as follows: let's suppose that there is a solution $\ell(r)$ finite at both singularities. We know that then the solution must satisfy $\ell(r_+) \neq 0$ and $\ell(\infty) = 0$ from the indicial equations at r_+ and ∞ and we can assume that $\ell(r_+) > 0$ without loss of generality. Now it will be more convenient to rewrite the second-order homogeneous equation (4.1) with $C = 0$ as follows

$$(r-r_+)(r-r_-)\ell'' + \left(r_+ + \frac{1-3\alpha^2}{1+\alpha^2}r_- - 2\frac{1-\alpha^2}{1+\alpha^2}\frac{r_+r_-}{r}\right)\ell' - 2\left(1 + \frac{2}{1+\alpha^2}\frac{r_+r_-}{r^2}\right)\ell = 0. \quad (4.6)$$

We easily see, from this equation, that ℓ' is positive at r_+ because the first term of the left hand side vanishes, the coefficient of ℓ' is positive there, and the coefficient of ℓ is negative definite. On the other hand, according to the maximum and minimum value theorem, ℓ must have a maximum value at a point r_m between r_+ and ∞. At the point $r = r_m$, ℓ' should be zero by our assumption. However, this leads us to conclude that ℓ'' should be

positive, and so $\ell(r)$ is downwardly convex at r_m. However, this contradicts our assumption that the function $\ell(r)$ has the maximum at r_m. Thus, we can conclude there can be no solution of the homogeneous equation (4.1) which is finite simultaneously at $r = r_+$ and ∞.

Let's suppose that $\ell_1(r)$ is such a solution of the homogeneous differential equation (4.1) which is finite at r_+ and divergent according to r^2 as $r \to \infty$. Then, the second solution $\ell_2(r)$ can be generated as

$$\ell_2(r) = \ell_1(r) \int_r^\infty \frac{R(r')^2}{r'^2 \lambda(r')^2 \ell_1(r')^2} \, dr'. \tag{4.7}$$

Evidently, $\ell_2(r)$ diverges by order $\ln(r - r_+)$ near $r = r_+$ because ℓ_1 is finite there, and ℓ_2 converges by order r^{-1} as r approaches ∞ because ℓ_1 diverges by order r^2 there. It will be sufficient to know these properties of the homogeneous solutions to extract qualitative aspects of the angular momentum due to electromagnetic fields.

On the other hand, for extremal black holes the indicial equation near $r = r_+$ differs from that above, and says that a solution goes to zero according to $(r-r_+)^{D_+}$ and the other solution diverges as $(r-r_+)^{D_-}$, where $D_\pm \equiv (3\alpha^2 - 1 \pm \sqrt{17\alpha^4 + 26\alpha^2 + 25})/2(1+\alpha^2)$. The indicial equation at ∞ is not different from that in the case of non-extremal black holes. Then, $\ell_1(r)$ is zero at r_+ and diverges at ∞, but $\ell_2(r)$ is divergent at r_+ and vanishes as $r \to \infty$.

5. Electromagnetic angular momentum

In this section we examine the angular momentum of an electric charge at rest in the field of a magnetically charged black hole using the properties of the solution shown in the previous section. In general, the total angular momentum J contains contributions from both the rotation of the black hole itself and the electromagnetic field around the black hole. There are additional electromagnetic fields induced by the rotation of the black hole besides the electric field of the charge and the magnetic field of the black hole. Those induced fields also contribute to the angular momentum of the system. However, we will separate out this contribution together with the intrinsic spin of the black hole from the total angular momentum leaving only the portion irrespective of the rotation of the black hole, *i.e.*, the 'electromagnetic angular momentum' coming from the electric field of the charge and the magnetic field of the black hole. It will be interesting to compare this electromagnetic angular momentum with the angular momentum of the charge-monopole system in flat space. Probably the easiest way to find the electromagnetic angular momentum is to compute the angular momentum for an electric charge at rest at a distance from a non-rotating magnetically charged black hole. However, it may not be possible to set up such a configuration from the outset, when the charge is held at a finite distance from the hole. The black hole may not stay static under the influence of the electric field of the charge, but it may keep on rotating as long as the charge is held at a fixed position forcefully.

To see this, let's suppose a magnetically charged black hole immersed in the weak, asymptotically uniform electric field. This situation is equivalent, by electromagnetic duality, to the case of an electrically charged black hole immersed in the weak, asymptotically

uniform magnetic field considered by Bičák and Dvořák in [9]. Therefore, we can directly write down the metric perturbation from their work. For the general stationary and axisymmetric perturbations, the only nonvanishing component of the metric has the form

$$g_{t\varphi} = -a \left(\frac{2M}{r} - \frac{Q^2}{r^2} \right) \sin^2 \theta + 2QE_0 \left(r - \frac{Q^2}{r} + \frac{Q^4}{2Mr^2} \right), \qquad (5.1)$$

where E_0 is the strength of the asymptotically uniform external electric field and a is a small angular momentum parameter. As mentioned in [9], this is not valid for arbitrarily large r since a uniform field in an asymptotically flat spacetime would contain an infinite amount of energy. It is meaningful only for r satisfying the condition $\mid rE_0 \mid \ll 1$. Setting $E_0 = 0$, the metric represents a slowly rotating Kerr-Newman black hole. With $a = 0$, it gives the perturbation describing the Reissner-Nordström black hole immersed in the weak, asymptotically uniform electric field of strength E_0. Using the formula (3.6), we may then read off the angular momentum within a sphere of radius r as

$$\mathcal{L}(r) = -a \left(M - \frac{2Q^2}{3r} \right) + QE_0 \left(Q^2 - \frac{r^2}{3} - \frac{2Q^4}{3Mr} \right). \qquad (5.2)$$

The first contribution explicitly comes from the rotation of the Kerr-Newman black hole and it would have have vanished if we had introduced the Reissner-Nordström black hole (of $a = 0$) from the outset. The second contribution comes from the electromagnetic fields of the system and vanishes only as the external electric field disappears. That is, it appears because the hole is immersed in the external electric field. When $a = 0$, the angular momentum within the horizon is given by

$$\mathcal{L}(r_+) = -\frac{2}{3}QE_0(M^2 - Q^2) \left(1 + \sqrt{1 - \frac{Q^2}{M^2}} \right), \qquad (5.3)$$

which is not zero unless the black hole approaches its extremality. This may originate from the electromagnetic fields inside the horizon, but may be perceived as usual rotation when viewed from the outside in accordance with the no-hair theorem. It is determined only by the charge and mass of the hole and the strength of the external field. It is thus tempting to conjecture that the magnetically charged black hole immersed in an external electric field should be constantly rotating with an angular momentum determined by those quantities. Let's suppose that one tries to stop the rotation of the hole by extracting its rotational energy (for example, via the Penrose process or the superradiant scattering) keeping the position of the charge fixed. However, during the extraction process, energy must be constantly supplied to the system through the external field when he tries to keep its strength constant. The rotation is kept constant, because the angular momentum is determined only by the charge and mass of the hole and the strength of the external field. Thus, it may not be realistic to consider a non-rotating magnetically charged black hole immersed in an external electric field.

On the same line, it may be impractical to consider a non-rotating magnetically charged black hole immersed in the electric field of the charge. Therefore, we have to consider first the system of a charge and a rotating magnetically charged black hole, and then disentangle

the effect due to the rotation of the hole itself leaving only the electromagnetic angular momentum. As a way to disentangle it from the angular momentum, we consider the angular momentum obtained by performing the Komar integral at the horizon: $\mathcal{L}_H \equiv \mathcal{L}(r_H)$, and then drop the contribution coming from this quantity to the total angular momentum measured in the asymptotic region: $J \equiv \mathcal{L}(\infty)$. To do this, we first consider the angular momentum of a slowly rotating dilaton black hole and examine the relation between the two quantities.

A. Angular momentum of a slowly rotating dilaton black hole. We first recall the angular momentum of a slowly rotating dilaton black hole mentioned in section 2. Inserting the metric component $g_{t\varphi}$ of (2.14) into the formula (3.6) and (3.7), we get

$$\mathcal{L}(r) = -\frac{a}{6}R^4(r)f'(r) = \frac{a}{6}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_- - \frac{4}{1+\alpha^2}\frac{r_+r_-}{r}\right). \tag{5.4}$$

The angular momentum is directly read by

$$J \equiv \mathcal{L}(\infty) = \frac{a}{6}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right), \tag{5.5}$$

which is coincident with the result obtained in [15]. Thanks to the formula (5.4), we can write the angular momentum contained between two spheres with radii r_1 and r_2 as the difference: $\mathcal{L}(r_2) - \mathcal{L}(r_1)$. We can then divide the angular momentum into two parts. The first part is the angular momentum inside the horizon. It may be interpreted as the intrinsic spin of the black hole and its magnitude is

$$\mathcal{L}_H \equiv \mathcal{L}(r_+) = \frac{a}{6}\left(3r_+ - r_-\right). \tag{5.6}$$

The second part is the angular momentum outside the horizon and is written by

$$\mathcal{L}_{\text{out}} \equiv \mathcal{L}(\infty) - \mathcal{L}(r_H) = \frac{2r_-a}{3(1+\alpha^2)} = \frac{2}{3}\frac{Q^2a}{r_+}. \tag{5.7}$$

This part can be regarded as the angular momentum stored in the electromagnetic field outside the black hole. Since there exists a non zero component of electric field induced due to the rotation of the black hole, the electromagnetic angular momentum is not zero as it would be for a non-rotating charged or a rotating neutral black hole.

Eliminating a by using (5.6) from (5.5), the angular momentum of a slowly rotating dilaton black hole can be rewritten in terms of the horizon angular momentum as

$$J = \frac{\mathcal{L}_H}{(3r_+ - r_-)}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right). \tag{5.8}$$

We will use this relation to decouple the effect of the spin of black hole from the electromagnetic field angular momentum below.

B. Angular momentum stored in the electrostatic and magnetostaic fields.
Here we construct an expression for the electromagnetic angular momentum using the properties, discussed in the previous section, of the particular and the homogeneous solutions of the second-order differential equation (4.1) in terms of ℓ_1, ℓ_2, and ℓ_p and their differentials. From the properties of the solutions, $\mathcal{L}(r)$ can be written as

$$\mathcal{L}(r) = k_1 \ell_1(r) + k_p \ell_p(r), \qquad r < b, \tag{5.9}$$

$$\mathcal{L}(r) = k_2 \ell_2(r) + \tilde{k}_p \ell_p(r), \qquad r > b, \tag{5.10}$$

where k_1, k_2, k_p, and $\tilde{k}_p$ are coefficients to be determined from matching conditions at $r = b$. The continuity of $\mathcal{L}$ and $d\mathcal{L}/dr$ at $r = b$ yields two conditions:

$$k_1 \ell_1(b) - k_2 \ell_2(b) + \bar{k}_p \ell_p(b) = 0, \tag{5.11}$$

$$k_1 \ell_1'(b) - k_2 \ell_2'(b) + \bar{k}_p \ell_p'(b) = 0, \tag{5.12}$$

where $\bar{k}_p \equiv k_p - \tilde{k}_p$. The discontinuity of $\mathcal{D}\mathcal{L}$ at $r = b$ yields that

$$\bar{k}_p \, \mathcal{D}\ell_p|_{r=b} = 2eQ, \tag{5.13}$$

where we used the fact that ℓ_1 and ℓ_2 are solutions of the homogeneous equation, i.e., $\mathcal{D}\ell_1(r) = \mathcal{D}\ell_2(r) = 0$. k_1, k_2, $\bar{k}_p$ are determined by above three conditions:

$$k_1 = -\frac{\ell_p(b)\ell_2'(b) - \ell_p'(b)\ell_2(b)}{\ell_1(b)\ell_2'(b) - \ell_1'(b)\ell_2(b)} \bar{k}_p \,, \tag{5.14}$$

$$k_2 = -\frac{\ell_p(b)\ell_1'(b) - \ell_p'(b)\ell_1(b)}{\ell_1(b)\ell_2'(b) - \ell_1'(b)\ell_2(b)} \bar{k}_p \,, \tag{5.15}$$

$$\bar{k}_p = -\frac{6eQ}{a}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right)^{-1}, \tag{5.16}$$

where we used (4.5) in the last line. From (4.4) and (5.10), it is straightforward to show that the total angular momentum is expressed in term of $\tilde{k}_p$ as follows

$$J = \lim_{r\to\infty}\mathcal{L}(r) = \tilde{k}_p \lim_{r\to\infty}\ell_p(r) = \frac{a}{6}\left(3r_+ + \frac{3-\alpha^2}{1+\alpha^2}r_-\right)\tilde{k}_p. \tag{5.17}$$

Therefore, we still need another additional condition to determine $\tilde{k}_p$. We may think that the total angular momentum is composed of two parts. The first part is the angular momentum due to the rotation of the black hole, which includes the intrinsic spin of the black hole and the contribution from magnetic and electric fields induced due to its rotation. The second part is the angular momentum stored in the electromagnetic field unaffected by the rotation of the black hole. This part corresponds to the angular momentum of an electric charge held at rest around a non-rotating magnetically charged black hole that would have if such configuration could been set up.

Since we are mainly interested in the latter, we separate out the former from the total angular momentum. To do this, we suppose that the angular momentum contained within the horizon is to be $\mathcal{L}_H = \mathcal{L}(r_+)$, that is,

$$\mathcal{L}_H = k_1 \ell_1(r_+) + k_p \ell_p(r_+). \tag{5.18}$$

The coefficient $\tilde{k}_p$ is determined from (5.14), (5.16), (5.18), and (4.4) and the angular momentum is given by the expression

$$J = \frac{\mathcal{L}_H}{(3r_+ - r_-)}\left(3r_+ + \frac{3 - \alpha^2}{1 + \alpha^2}r_-\right) + eQ\left(1 - \frac{\ell_1(r_+)}{\ell_p(r_+)}\frac{\ell_p(b)\ell_2'(b) - \ell_p'(b)\ell_2(b)}{\ell_1(b)\ell_2'(b) - \ell_1'(b)\ell_2(b)}\right). \quad (5.19)$$

The first term of the right hand side represents the part due to the rotation of the black hole as discussed in the previous subsection. The second term is our main result of this subsection and represents the angular momentum stored in the electromagnetic field which are not affected by the spin of black hole. The electromagnetic angular momentum J_{em} can be rewritten as follows

$$J_{em} = eQ\left[1 - \frac{\ell_1(r_+)}{\ell_p(r_+)}\left(\frac{\ell_p(b)}{\ell_1(b)} - \frac{b^2\lambda^2(b)}{R^2(b)}\left(\ell_p(b)\ell_1'(b) - \ell_p'(b)\ell_1(b)\right)\frac{\ell_2(b)}{\ell_1(b)}\right)\right], \quad (5.20)$$

where we have used the relation (4.7) and eliminated $\ell_2'(b)$. Note that this quantity is independent of the angular momentum parameter a, even though we introduced it into the inhomogeneous solution ℓ_p.

Unfortunately, exact forms of ℓ_1 and ℓ_2 are not known for general coupling α. We know explicit forms of them only for $\alpha = 0$ and $\sqrt{3}$. Therefore, we first examine quantitatively the electromagnetic angular momentum for $\alpha = 0$ and $\sqrt{3}$ with explicit forms of ℓ_1 and ℓ_2. We then qualitatively estimate the behavior of J_{em} using the properties of ℓ_1, ℓ_2, and ℓ_p and confirm this estimation by analyzing the behavior of $\mathcal{L}(r)$ for arbitrary coupling α. Finally, we will numerically find the angular momentums for values $\alpha = 1/\sqrt{3}$ and 1. We choose these values because the values $\alpha = 0$, $1/\sqrt{3}$, 1, $\sqrt{3}$ are well within regimes of α where the extremal black holes show qualitatively different behaviors. Moreover, they are meaningful in the context of supergravity and string theories. As mentioned in section 2, when $\alpha = 0$, the action (2.1) is the Einstein-Maxwell system with an uncoupled scalar, which we can take to be constant. The theory with $\alpha = 1$ arises naturally in the reduction of the heterotic string on T^6 [17]. The value $\sqrt{3}$ appears in Kaluza-Klein compactification from five to four dimensions. Finally, when $\alpha = 1/\sqrt{3}$, the four-dimensional extreme dilaton black hole is interpreted as the double-dimensional reduction of a nonsingular five-dimensional black string which is a solution of the pure five-dimensional supergravity [18].

C. For $\alpha = 0$. When $\alpha = 0$, two independent solutions of the homogeneous equation are given by

$$\ell_1(r) = r^2 - 3r_+ r_- + \frac{4r_+^2 r_-^2}{r_+ + r_-}\frac{1}{r}, \quad (5.21)$$

$$\begin{aligned}
\ell_2(r) &= \ell_1(r)\int_r^\infty \frac{dr'}{\lambda^2(r')\ell_1^2(r')} \\
&= -\frac{(r_+ + r_-)^2}{(r_+ - r_-)^4}\left[r + \frac{1}{2}(r_+ + r_-) - \frac{2}{3}\frac{r_+ r_-(r_+^2 + 10r_+ r_- + r_-^2)}{(r_+ + r_-)^2}\frac{1}{r}\right] \\
&\quad - \frac{(r_+ + r_-)^2}{(r_+ - r_-)^5}\left[r^2 - 3r_+ r_- + \frac{4r_+^2 r_-^2}{r_+ + r_-}\frac{1}{r}\right]\ln\left(\frac{r - r_+}{r - r_-}\right), \quad (5.22)
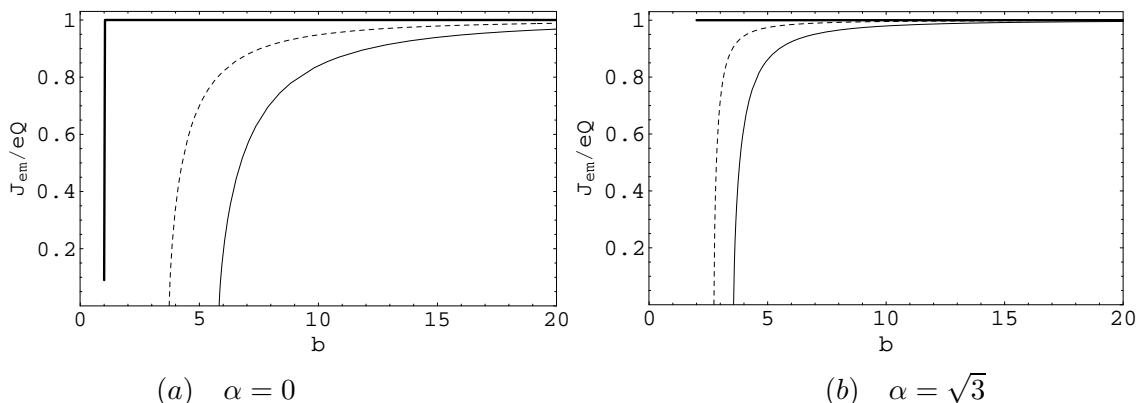\end{aligned}$$

Figure 1: J_{em}/eQ with respect to b when $e = 0.001$ and $Q = 1$ for $\alpha = 0$ and $\sqrt{3}$. The thick solid line is for an extremal black hole with the critical mass $M_c = Q/\sqrt{1 + \alpha^2}$, the dashed line and the thin solid line for a black hole with $M = 2M_c$ and $3M_c$, respectively.

and the inhomogeneous solution of eq. (4.1) is given by

$$\ell_p(r) = \frac{a}{6}\left(3(r_+ + r_-) - \frac{4r_+ r_-}{r}\right), \tag{5.23}$$

with $C = -a(r_+ + r_-)$. We can easily check that all the properties mentioned in the previous section hold with these explicit forms of solutions. ℓ_1 is finite at $r = r_+$ and $\ell_1(r_+) = r_+(r_+ - r_-)^2/(r_+ + r_-)$, while it is divergent according to r^2 as $r \to \infty$. ℓ_2 diverges in the order of $\ln(r - r_+)$ at $r = r_+$, and it goes to zero according to r^{-1} as $r \to \infty$. On the other hand, in the case of the extremal black hole, ℓ_1 vanishes by order $(r - r_+)^2$, while ℓ_2 diverges by order $(r - r_+)^{-3}$ at $r = r_+$.

When we insert these solutions into (5.20), the electromagnetic angular momentum is calculated as follows

$$J_{em} = eQ\left[1 - \frac{\ell_1(r_+)}{\ell_p(r_+)}\left(\frac{\ell_p(b)}{\ell_1(b)} - a(r_+ + r_-)\left(b - 2\frac{r_+ r_-}{r_+ + r_-}\right)\lambda^2(b)\frac{\ell_2(b)}{\ell_1(b)}\right)\right]. \tag{5.24}$$

We can easily check that this result is coincident with the result obtained for the Reissner-Nordström black hole in [12]. The angular momentum J_{em} changes from 0 to eQ as b goes from r_+ to ∞. Furthermore, by differentiating (5.24) with respect to b one can easily show that J_{em} is a monotonic function of b. That is, J_{em} changes monotonically from 0 to eQ. In the case of extremal black holes, $\ell_1(r_+) = 0$ and the second term in the square bracket of (5.24) vanishes identically. The angular momentum is then precisely eQ regardless of the separation distance b between the charge and the black hole. Plots for J_{em} of (5.24) with respect to b are shown in figure 1. (a).

D. For $\alpha = \sqrt{3}$. When $\alpha = \sqrt{3}$, two independent solutions of the homogeneous equation

are as follows:

$$\ell_1(r) = \frac{1}{r}(r - r_-)^3, \tag{5.25}$$

$$\ell_2(r) = \ell_1(r) \int_r^\infty \frac{R^2(r')}{r'^2 \lambda^2(r') \ell_1^2(r')} dr'$$

$$= -\frac{r_+}{(r_+ - r_-)^3} \left[r + \frac{1}{2}(r_+ - 5r_-) - \frac{r_-}{6r_+} \left(r_+^2 - 5r_+ r_- - 2r_-^2 \right) \frac{1}{r} \right]$$

$$- \frac{r_+}{(r_+ - r_-)^4} \frac{(r - r_-)^3}{r} \ln \left(\frac{r - r_+}{r - r_-} \right), \tag{5.26}$$

The inhomogeneous solution follows from (4.4) as

$$\ell_p(r) = \frac{a}{6} \left(3r_+ - \frac{r_+ r_-}{r} \right), \tag{5.27}$$

with a constant nonhomogeneous term $C = -ar_+$. Evidently, these solutions also satisfy all the properties mentioned in the previous section as for $\alpha = 0$: $\ell_1(r_+) = (r_+ - r_-)^3/r_+$ at $r = r_+$ and diverges according to r^2 as $r \to \infty$. ℓ_2 diverges in the order of $\ln(r - r_+)$ at $r = r_+$, and it goes to zero according to r^{-1} as $r \to \infty$. On the other hand, in the case of the extremal black holes, ℓ_1 vanishes by order $(r - r_+)^3$, while ℓ_2 diverges by order $(r - r_+)^{-1}$ at $r = r_+$.

Inserting these solutions into (5.20), we obtain the following expression for the electromagnetic angular momentum:

$$J_{em} = eQ \left[1 - \frac{\ell_1(r_+)}{\ell_p(r_+)} \left(\frac{\ell_p(b)}{\ell_1(b)} - ar_+(b - r_+) \frac{\ell_2(b)}{\ell_1(b)} \right) \right]. \tag{5.28}$$

We can easily check that, as for $\alpha = 0$, J_{em} changes monotonically from 0 to eQ as b goes from r_+ to ∞ for non-extremal black holes and is eQ regardless of the separation distance b for extremal black holes. Plots for J_{em} of (5.28) with respect to b are shown in figure 1. (b).

E. For arbitrary α. Even though we don't have explicit forms of homogeneous solutions ℓ_1 and ℓ_2, we can qualitatively estimate the electromagnetic angular momentum from (5.20) using the properties of the solutions discussed in the previous section. First of all, as for the case $\alpha = 0$, $\sqrt{3}$, the angular momentum depends upon the separation distance b between the black hole and the electric charge. Since ℓ_2 converges by order r^{-1} and ℓ_1 diverges by order r^2 as r approaches ∞, the second term vanishes as the separation distance goes to infinity. That is, (5.20) says that, as $b \to \infty$, the electromagnetic angular momentum approaches eQ. On the other hand, when the electric charge approaches the horizon, the angular momentum becomes zero, *i.e.*, $J_{em} \to 0$ as $b \to r_+$. This is easily seen from observation that the second term in the large round bracket vanishes as $b \to r_+$. It comes from the facts that ℓ_1 goes to a finite value, ℓ_2 diverges only by order $\ln(r - r_+)$, but λ^2 vanishes by order $(r - r_+)$ as $r \to r_+$. Thus the angular momentum J_{em} changes from 0 to eQ as b goes from r_+ to ∞.

In the case of extremal black holes, i.e., when $r_+ = r_-$, ℓ_1 goes to zero by order larger than $(r - r_+)^2$ near the horizon as mentioned above, and ℓ_1 vanishes identically at $r = r_+$.
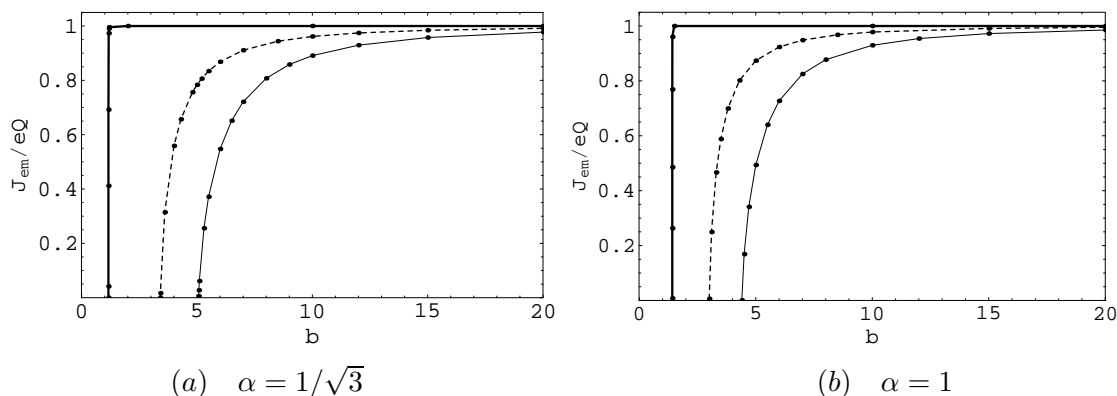
Figure 2: J_{em}/eQ with respect to b when $e = 0.001$ and $Q = 1$ for $\alpha = 1/\sqrt{3}$ and 1. The thick solid line is for an extremal black hole with $M_c = Q/\sqrt{1 + \alpha^2}$, the dashed line and the thin solid line for a black hole with $M = 2M_c$ and $3M_c$, respectively.

The second term in the square bracket then vanishes also. Thus, the angular momentum is precisely eQ regardless of the separation b between the charge and the dilaton black hole, as for the monopole-charge system in flat space.

These properties are confirmed numerically. The electromagnetic angular momenta J_{em}/eQ are plotted versus the separation distance b in figure 2 for extremal and non-extremal black holes. The numerical results for $\alpha = 1/\sqrt{3}$ and 1 has similar behavior as for $\alpha = 0, \sqrt{3}$.

So far we have examined the angular momentum of an electric charge placed at rest outside a magnetically charged dilaton black hole. We have seen that the electromagnetic angular momentum, the part independent of the rotation of the black hole, shows several common features regardless of the dilaton coupling strength, even though the qualitative features of black holes are drastically different depending on the dilaton coupling. It changes monotonically from eQ to 0 as the charge goes down from infinity to the horizon. However, in the extremal black hole case, it is precisely eQ as in the charge-monopole system in flat space and its dependence upon the separation distance disappears.

Before ending this section, we would like to see what happens when the charge approaches the horizon from infinity. Let's suppose the black hole does not rotate at the outset and the charge is placed at rest at an infinite distance from it. Then the angular momentum of the system is all stored in the electromagnetic field outside the horizon and its magnitude is simply eQ. As the charge moves down radially toward the horizon, the black hole starts to rotate and the spin angular momentum J_s increases monotonically because the total angular momentum J is conserved, while the electromagnetic angular momentum J_{em} decreases monotonically. When the charge reaches the horizon, the transfer is completed, i.e., J_{em} is decreased in magnitude from eQ to zero, while J_s is increased from zero to eQ. As a result, the magnetically charged black hole turns into a dyon black hole charged with e and Q and rotating with the total angular momentum of eQ. On the other hand, for the extreme black hole $J_{em} = eQ$ for all separation distance between the

charge and black hole. Thus as the charge moves down toward the black hole, the angular momentum transfer does not occur and the black hole does not spin up.

6. Discussions

We have analyzed the angular momentum of an electric charge placed at rest outside a magnetically charged dilaton black hole. Even though the qualitative features of black holes depend crucially on the dilaton coupling, the angular momentum stored in the electromagnetic fields shows several common features regardless of the strength of dilaton coupling. It changes monotonically from eQ to 0 as the charge goes down from infinity to the horizon. Therefore, one can spin up a magnetically charged black hole by throwing an electric charge radially into it. However, in the extremal black hole case, its dependence upon the separation disappears and the field angular momentum becomes independent of the distance between the two objects. That is, it is precisely eQ as for the system of an electric charge and a magnetic monopole in flat space, so spinning up the black hole is not possible as long as the charge stays outside the horizon as pointed out in [12]. Explicitly, these results should equally hold for a system of a magnetic monopole and an electrically charged black hole by the electromagnetic duality.

Clearly these features raise several questions. Why does the field angular momentum change as the separation varies? That is, why is it eQ as the charge rests at infinity? Why is it zero as the charge reaches the horizon? Why does its magnitude remain frozen with the value eQ in the extremal limit of black holes independently of the separation distance? Finally, can we expect those things to hold for other systems including different magnetically charged black holes in various theories with more complicated field contents? It seems to be natural to ask this question, because those phenomena are common to all dilaton black holes which have drastically different features depending on the dilaton coupling. Discussions on these questions are presented below.

At first glance, one may think that changing of the electromagnetic angular momentum upon the separation distance occurs only in the presence of gravity, because the electromagnetic angular momentum of the charge-monopole system is given by the product of the two charges irrespective of the separation distance in the absence of gravity. However, it is not true. The exotic result in flat space comes from only a simple consideration that both electric and magnetic charges are point-like. The contribution of the electromagnetic field to the angular momentum is directly calculated from

$$\vec{J}_{em} = \int d^3x \; \vec{r} \times (\vec{E} \times \vec{B}). \tag{6.1}$$

With the fields of point-like electric charge and monopole given by $\vec{E} = e\vec{r'}/r'^3$ and $\vec{B} = Q\vec{r}/r^3$ with $r' \equiv |\vec{r} - \vec{b}|$ respectively, the angular momentum density, integrand of (6.1), varies according to the separation distance between the two objects, but the integral of the density over the whole region is independent of the separation distance. There is no way to transfer the electromagnetic angular momentum to anything else. However, if we introduce a magnetically charged classical object (with a nontrivial magnetic charge distribution)

instead of the point-like monopole, then we may observe the separation dependence of the electromagnetic angular momentum even in flat space.

To see this, we address a simple example in flat space: a system of an electric charge and a magnetically charged non-conducting spherical shell of radius R. The magnetic charges are evenly distributed on the shell and the electric charge is located at a distance b apart from the center of the shell. The contribution to the integral comes from only fields outside the shell because the magnetic field vanishes inside the shell. When both objects are at rest, the electromagnetic angular momentum is then directly calculated to give

$$J_{em} = eQ \left(1 - \frac{R^2}{3b^2} \right) \quad \text{for} \ \ b \geq R, \tag{6.2}$$

$$= eQ \frac{2b}{3R} \qquad \qquad \text{for} \ \ b < R. \tag{6.3}$$

Certainly, the magnitude of the electromganetic angular momentum varies with the separation distance: it changes from 0 to eQ as the electric charge moves from the center to infinity. This can be explained as follows. When the charge is placed at infinity from the shell, i.e., when $b \gg R$, the electric field flux across it vanishes. This is the same as letting $R \to 0$ while keeping b finite. Then the angular momentum is eQ and is all stored in the electromagnetic fields outside the horizon. When the electric charge is closer to the shell, the electric field flux crossing the shell is larger. Then, the integration (6.1) is smaller because the integral is restricted outside the shell, unlike the charge and monopole system. Finally, when the electric charge reaches the center of the shell, the density $\vec{r} \times (\vec{E} \times \vec{B})$ becomes zero over the whole region because both fields are purely radial from the center of the shell. Of course, in a dynamical process the total angular momentum should be conserved. As the electric charge moves toward the shell, the latter is forced to rotate about its axis due to the Lorentz force exerted on magnetic charges by the electric field crossing it, and hence becomes spinning. Once the shell starts to rotate, an electric dipole field is induced around it due to magnetic charges rotating about the symmetric axis and a part of the angular momentum is stored in this dipole field and the existing magnetic field. That is, as the electric charge gets closer, the electromagnetic angular momentum is transferred to the spin of the shell and stored in the magnetic field and the induced electric dipole field. The transfer becomes complete as the electric charge reaches the center of the sphere. On the other hand, the electromagnetic angular momentum becomes independent of the separation distance as the shell shrinks to a point, *i.e*, as $R \to 0$. This confirms that the change of the electromagnetic angular momentum upon the separation distance comes from the nontrivial magnetic charge distribution.

Another interesting example is a system of an electric charge and a neutral conducting sphere in which a magnetic monopole is centered. In the presence of the charge, two image charges can be introduced within the conducting sphere, the first $q' = e(R/b)$ at the center and the second $q'' = -e(R/b)$ at a distance $b'' = R^2/b$ from the center. The first image charge does not contribute to the angular momentum because its electric field is purely radial like the magnetic field of the monopole. The second contributes negatively because its charge is opposite to that of the real charge. The image charge is larger as the charge is

closer to the conducting sphere. When the electric charge contacts the sphere finally, the second image charge becomes $-e$ and offsets the real charge leaving only the first image charge of e at the center. Then the electromagnetic angular momentum vanishes because both electric and magnetic fields are purely radial. On the contrary, when the charge goes to infinity from the sphere, the image charges are diminished and the electric field flux crossing the sphere becomes negligible, and so the electromagnetic angular momentum is all stored in fields outside the sphere and has the magnitude of eQ. Taking into account the contribution of image charges and using the results of the previous example, we obtain the electromagnetic angular momentum for this system

$$J_{em} = eQ \left(1 - \frac{R^2}{b^2} \right) \quad \text{for} \quad b \geq R, \tag{6.4}$$

which, as expected, runs monotonically from 0 to eQ as the electric charge goes from the sphere to infinity. This is very similar to the result obtained for the system of an electric charge and a magnetically charged black hole. Again, as $R \to 0$, the electromagnetic angular momentum becomes independent of the separation distance, showing that its change upon the separation comes from the nontrivial electromagnetic structure surrounding the monopole.

Let's return to the charge and black hole system. The result obtained in the previous sections for the system can be explained analogously to the above examples. In fact, the horizon of a black hole plays a role very similar to that of the conducting sphere in the above example, even if a black hole has no material surface that differs from the surrounding space in contrast to the conducting sphere. As is well known in the black hole electrodynamics, for a distant observer the horizon can be thought of as a fictitious surface having an electric charge density that compensates for the flux of electric field across the horizon, and an electric current that closes tangential components of the magnetic fields at the horizon [20]. Thus, the separation dependence of the electromagnetic angular momentum of an electric charge and a magnetically charged black hole could be explained in an analogous way to the above example introducing a charge distribution on or image charge within the horizon, though the electromagnetic fields are distorted and more complicated due to the gravity around the black hole.

This can be interpreted from another perspective. In the previous sections we have obtained the monotonic change of the angular momentum upon the separation, only solving the equations of motion (2.2)–(2.4) without introducing *ad hoc* electromagnetic properties of the horizon. Even though a black hole simply has a point source at the origin and has no material structure, it has a nontrivial spacetime structure and is shielded by the event horizon. The horizon does such a role as the conducting shell of the above example does. Of course, there are no real material surface and no real charge distribution at the horizon. However, the event horizon makes the electromagnetic angular momentum across it indistinguishable from the intrinsic spin of the black hole in accordance with the no hair theorem.

What happens is as follows. When the charge is placed at rest at infinity from a non-rotating charged black hole, no flux of the electric field passes through the black hole and

so no flux of the electromagnetic angular momentum crosses it. The integral of (3.4) on the horizon, $\mathcal{L}(r_+)$, then vanishes. The angular momentum is all stored in the electromagnetic fields outside the horizon and its value is then eQ as for the charge and monopole system in flat space.

On the other hand, when the charge approaches the black hole, the electric field lines passing through the black hole increases and also the flux of the electromagnetic field angular momentum across the horizon increases. Once the electromagnetic angular momentum crosses over the horizon, it becomes indistinguishable from the spin of the black hole according to the no hair theorem. That means the influx of the electromagnetic angular momentum across the horizon appears as the intrinsic spin of the black hole. As the electric charge gets closer to the black hole, the flux of the electromagnetic angular momentum across the horizon increases and the spin of the black hole becomes larger. If the black hole rotates, the existing magnetic and electric fields are supplemented with induced electric and magnetic fields respectively, due to the dragging of the inertial frame into the rotational motion around the black hole. These induced fields also store an amount of electromagnetic angular momentum, which depends on the rotation of the black hole and vanishes only when the angular momentum parameter a of the black hole becomes zero. Therefore, as the charge approaches the black hole, the electromagnetic angular momentum in the existing fields is continuously transferred to the spin angular momentum and the electromagnetic angular momentum in the induced fields.

When the electric charge reaches the horizon, all the electrostatic multipoles except the monopole fade away in accordance with the no hair conjecture. The electrostatic monopole field due to this charge then becomes purely radial like the magnetic monopole field of the black hole. So the angular momentum stored in these monopole fields is zero and the transfer gets complete. A part of it is transferred to the spin of the black hole and the rest is stored in the induced fields. Once the charge reaches the horizon, then the exterior fields are exactly those of the rotating dyonic dilaton black hole with both the electric and magnetic charges. For a slowly rotating dyonic dilaton black hole, the metric can be displayed by (2.14) replacing Q^2 with $Q^2 + e^2$ [21]. Since the total angular momentum is eQ, the angular momentum parameter a then can be written, from (2.17), as

$$a = 6eQ \left(3\bar{r}_+ + \frac{3 - \alpha^2}{1 + \alpha^2} \bar{r}_- \right)^{-1}, \tag{6.5}$$

where $\bar{r}_\pm$ are positions of outer and inner horizons of the dyonic dilaton black hole and are obtained from (2.10) and (2.11) by replacing Q^2 with $Q^2 + e^2$. Then, from (5.6) and (5.7), the angular momentum within the horizon is given by

$$\mathcal{L}_H = eQ \frac{(1 + \alpha^2)(3\bar{r}_+ - \bar{r}_-)}{3(1 + \alpha^2)\bar{r}_+ + (3 - \alpha^2)\bar{r}_-}, \tag{6.6}$$

and the angular momentum in the electromagnetic field induced by the rotation of the black hole is

$$\mathcal{L}_{out} = eQ \frac{4\bar{r}_-}{3(1 + \alpha^2)\bar{r}_+ + (3 - \alpha^2)\bar{r}_-}. \tag{6.7}$$

From above arguments, we can conclude that the separation dependence of the electromagnetic angular momentum of the charge and black hole system originates from the existence of the horizon, which serves as a nontrivial spacetime structure and obliterates all details of the angular momentum swallowed up in accordance with the no hair theorem.

Finally, let's see what happens as the black hole is extremal from the outset. The results of previous sections say that the angular momentum stored in the electromagnetic field outside the horizon is precisely eQ and is independent of the separation distance between the charge and the black hole. It seems to be easy to give an explanation on this phenomenon for the dilaton black hole. As mentioned in section 2, when $\alpha > 0$, the proper size of the extremal black hole is zero in that the area of the horizon $4\pi R^2(r_+)$ vanishes and the degenerate horizon at $r = r_+$ is singular. The extremal black hole looks like a point-like object. The situation is then the same as with the charge-monopole system in flat space. The electric field flux passing the black hole is zero. The angular momentum is all stored in the electromagnetic field outside the horizon and has the constant value eQ.

This kind of explanation does not seem to hold for the system of the electric charge and the extremal Reissner-Nordström black hole (as $\alpha = 0$), because its horizon area is $4\pi r_+^2$ and its proper size is not zero. We may then naively expect that its cross-sectional area for the capture of the electric flux should be πr_+^2 and the electric flux across the hole should be non-vanishing. According to our naive expectation, the contribution to the total angular momentum of the system from behind the horizon should be non-zero in the extremal limit in this configuration. However, this contradicts the above result that the electromagnetic angular momentum outside the Reissner-Nordström black hole is precisely eQ, independently of the separation distance between the charge and the black hole.

The puzzle is solved when we invoke that the component of the electromagnetic fields normal to the horizon tends to be zero and the flux lines are totally expelled, as the black hole approaches extremality. This is analogous to the "Meissner effect" for a magnetic field around a superconductor. This phenomenon was first pointed out by J. Bičák and Dbořák [9] in the context of the Einstein-Maxwell theory. They considered the electrically charged Reissner-Nordström black hole immersed in the weak, asymptotically uniform magnetic field and found that the magnetic flux lines are expelled from the black hole in the extremal limit. Later, it was shown by Chamblin, Emparan and Gibbons [10] that this effect broadly exists for various extremal black hole solutions including the dilaton black hole and other extremal solitonic objects (such as p-branes) in string theory and Kaluza-Klein theory.

Due to the electromagnetic duality, we can expect that the electric field is expelled from the magnetically charged black hole in the extremal limit. Then the effective cross-sectional area for the capture of the electric flux shrinks to zero and the electric flux across the hole vanishes. The contribution to the total angular momentum of the system from behind the horizon is then zero, because the black hole is non-rotating from the outset and the electric field inside the horizon does not exist. Thus, even though the proper size of the horizon is not zero in the extremal limit, the angular momentum is all stored in the electromagnetic field outside the horizon as in the case of the charge-monopole system in flat space and is precisely eQ.

On the other hand, we could give an analogous explanation to the system of the charge and the extremal dilaton black hole (as $\alpha > 0$). As is well known, in four dimensions the gauge field equation is conformally invariant. This means that, when we deal with the dilaton black hole, the electromagnetic field does not distinguish whether we are working in the Einstein frame or any other conformal frame related to it by an overall rescaling of the metric by a factor of the dilaton as mentioned in [10]. In this sense, we could just as well work in a conformally related metric where the extremal horizon is non-singular. From the dilaton black hole solution (2.5)–(2.7), we easily see that such a conformal frame is $e^{2\alpha\phi}ds^2$. In this frame the extremal black hole area is equal to $4\pi r_+^2$. However, the electric flux across the horizon vanishes as shown in [10]. Once again, for the system of the charge and the extremal dilaton black hole the contribution to the angular momentum from inside the horizon is zero, because the black hole is non-rotating from the outset and the electric field is absent inside the horizon. As a result, the field flux expulsion from the extremal black hole horizon leads to the phenomenon that the electromagnetic angular momentum of the system comprising an electric charge and an extremal magnetically charged black hole is independent of the separation distance between the two objects and precisely eQ as for the charge-monopole system in flat space.

In conclusion, above arguments appear to settle all the questions in the second paragraph of this section. The reason why the electromagnetic angular momentum changes from eQ to 0 as the charge is brought down from infinity to the horizon is due to the no-hair nature of black holes in that the horizon makes the electromagnetic field angular momentum across it indistinguishable from the intrinsic spin of the black hole and it makes the electric field purely radial (leaving only the monopole field) when the charge reaches the horizon. Next, the electromagnetic angular momentum having the precise value eQ in the extremal limit of black holes independently of the separation distance is essentially related to the phenomenon that the electric flux lines are expelled from extremal black holes. Since both the no-hair nature of black holes and the field expulsion from extremal black holes are generic for black holes, we expect aforementioned properties of the electromagnetic angular momentum to be generic for systems of an electric charge and a magnetically charged black hole (or a magnetic monopole and an electrically charged black hole, by the electromagnetic duality) in various theories with more complicated field contents, namely, in string theory and Kaluza-Klein theory.

Acknowledgments

References

[1] R.S. Hanni and R. Ruffini, *Lines of force of a point charge near a schwarzschild black hole*, *Phys. Rev.* **D 8** (1973) 3259.

[2] J.M. Cohen and R.M. Wald, *Point charge in the vicinity of a Scwarzschild black hole*, *J. Math. Phys.* **12** (1971) 1845.

[3] B. Linet, *Electrostatics and magnetostatics in the Scwarzschild metric*, *J. Phys.* **A 9** (1976) 1081.

[4] B. Leaute and B. Linet, *Electrostatics in a Reissner-Nordstrom space-time*, *Phys. Lett.* **A 58** (1976) 5.

[5] D. Bini, A. Geralico and R. Ruffini, *On the equilibrium of a charged massive particle in the field of a Reissner-Nordstroem black hole*, *Phys. Lett.* **A 360** (2007) 515 [gr-qc/0608139].

[6] D. Bini, A. Geralico and R. Ruffini, *Charged massive particle at rest in the field of a Reissner-Nordstroem black hole*, *Phys. Rev.* **D 75** (2007) 044012 [gr-qc/0609041].

[7] F.J. Zerilli, *Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordstroem geometry*, *Phys. Rev.* **D 9** (1974) 860.

[8] R.M. Wald, *Black hole in a uniform magnetic field*, *Phys. Rev.* **D 10** (1974) 1680.

[9] J. Bicak and L. Dvorak, *Stationary electromagnetic fields around black holes. 3. General solutions and the fields of current loops near the Reissner-Nordstrom black hole*, *Phys. Rev.* **D 22** (1980) 2933.

[10] A. Chamblin, R. Emparan and G.W. Gibbons, *Superconducting p-branes and extremal black holes*, *Phys. Rev.* **D 58** (1998) 084009 [hep-th/9806017].

[11] J. Jackson, *Classical electrodynamics*, Wiley, New York (1975).

[12] D. Garfinkle and S.-J. Rey, *Angular momentum of an electric charge and magnetically charged black hole*, *Phys. Lett.* **B 257** (1991) 158.

[13] G.W. Gibbons and K.-I. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, *Nucl. Phys.* **B 298** (1988) 741.

[14] D. Garfinkle, G.T. Horowitz and A. Strominger, *Charged black holes in string theory*, *Phys. Rev.* **D 43** (1991) 3140.

[15] J.H. Horne and G.T. Horowitz, *Rotating dilaton black holes*, *Phys. Rev.* **D 46** (1992) 1340 [hep-th/9203083].

[16] C.F.E. Holzhey and F. Wilczek, *Black holes as elementary particles*, *Nucl. Phys.* **B 380** (1992) 447 [hep-th/9202014].

[17] T. Ortín, *Gravity and strings*, Cambridge University Press (2004).

[18] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Higher dimensional resolution of dilatonic black hole singularities*, *Class. and Quant. Grav.* **12** (1995) 297 [hep-th/9410073].

[19] R.M. Wald, *General relativity*, University of Chicago Press (1984).

[20] I.D. Novikov and V.P. Frolov, *Physics of black holes*, Kluwer Academic Publishers (1989).

[21] M. Kasuya, *Exact solution of a rotating dyon black hole*, *Phys. Rev.* **D 25** (1982) 995.

[22] C. Bunster and M. Henneaux, *A monopole near a black hole*, hep-th/0703155.